Build an early foundation for algebra success

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R&D

Research tells us that success in algebra is a factor in many other important student outcomes. Emerging research suggests that students who start an algebra curriculum in the early grades take to the subject better in secondary school.

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By Eric Knuth, Ana Stephens, Maria Blanton, and Angela Gardiner

Mathematics education in K-12 is a perennial national concern. Responses have ranged from alarmist to calls for curricular reform. The common thread is that students must acquire greater knowledge of and improve performance in mathematics. One approach to improving students' mathematical knowledge and performance is to build a foundation for their success in algebra.

Why algebra? Scholars say algebra is the linchpin to success in mathematics because of its foundational role in all areas of mathematics (NCTM, 2000; National Mathematics Advisory Panel, 2008; RAND Mathematics Study Panel, 2003). Algebra provides the mathematical tools to represent and analyze quantitative relationships, to model situations, and to solve problems in every mathematical domain. Algebra also is widely recognized as a gatekeeper to future educational and employment opportunities (Kaput, 1998; Moses & Cobb, 2001; National Research Council, 1998). Though two decades old, Schoenfeld's (1995) statement still holds true today:

> Algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often to even undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society (pp. 11-12).

Algebra's gatekeeper effect is also evident in many state policies in which substantial proficiency in algebra is required for graduation.

Yet, despite the importance of and attention placed on algebra, students have been largely unsuccessful developing a deep understanding of this mathematical domain with the traditional arithmetic-then-algebra approach: arithmetic in the elementary grades followed by algebra in the secondary grades. This approach has not enabled students to successfully navigate the transition from concrete, arithmetic reasoning to the increasingly complex, abstract algebraic reasoning required for secondary school mathematics and beyond. As a result, it is now widely accepted that algebra should play a role in school mathematics at all grade levels, beginning in the elementary grades. And, in fact, introducing early algebra in elementary school is viewed as the most critical factor for students' long-term success in algebra (Katz, 2007).

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Early algebra sets foundation for success

Early algebra does not mean algebra earlier, as in moving content from secondary algebra courses into the elementary grades. Instead, it means laying a foundation for developing understanding of such content by building elementary students' natural, informal intuitions about patterns, relationships, and structure into formalized ways of mathematical thinking. Early algebra does not mean replacing traditional arithmetic content with algebra content. Instead, it means extending the arithmetic typically taught in elementary school so that young children learn to see and reason with its underlying structure and properties and develop the ability to identify, describe, and analyze how quantities vary in relation to each other. Here are three core understandings critical to success in algebra accompanied by classroom activities that can help students build these understandings:

#1. The ability to use variables to represent unknowns or varying quantities is critical to success in algebra. Traditional elementary school mathematics, however, is computation-based and answer-oriented and does not typically focus on regularities found in computational work or their representations. Consider, on the other hand, the early algebra activity in Figure 1 that is focused on students' identification and representation of a fundamental arithmetic property used in their computational work

In this activity, students explore the Commutative Property by examining true/false equations. Some students may approach these equations by computing the sum on each side of the equation and then comparing the sums (revealing a lack of attention to the underlying structure of the equation), while other students will notice the underlying structure and recognize immediately that the equations are true. In the latter case, these students can be encouraged to express verbally what they notice in these equations — that, "when you add two numbers, the order doesn't matter." Teachers often assume students will have difficulty representing computational regularities using variables, and this is often the case if students are asked to represent foreign ideas. However, when teachers ask students to represent relationships that *already make sense to them*, the transition from words to variables actually is not as difficult as might be expected. In this case, representing the Commutative Property as $a + b = b + a$ is not such a big leap.

While the Commutative Property provides the context for this activity, the activity is about far more than having students understand the property, since students probably already understand it intuitively. The activity provides a familiar context for students to engage in the algebraic work of identifying regularities in computation, making generalizations about the regularities, and using symbols to represent those generalizations.

#2. A core understanding critical to student success in algebra is that the equal sign represents a relation between two equivalent quantities. Adults might assume that understanding the meaning of the equal sign is fairly straightforward and that students

FIGURE 1.
Generalizing and representing a computational regularity using variables

A. Which of the following number sentences are true? Use numbers, pictures, or words to explain your reasoning.

$17 + 5 = 5 + 17$

$20 + 15 = 15 + 20$

$148 + 93 = 93 + 148$

B. What numbers make the following number sentences true?

$25 + 10 =$ ___ $+ 25$

___ $+ 237 = 237 + 395$

$38 +$ ___ $=$ ___ $+ 38$

C. What do you notice about these problems? What can you say about the order in which you add two numbers? Describe your conjecture in words.

D. Represent your conjecture using variables. ***[Students will have been introduced to notion of variable in prior lessons.]***

FIGURE 2.
Developing a relational view of the equal sign

A. Which of the following equations are true? Explain.

$4 + 6 = 10$	$2 + 3 = 5 + 4$
$4 + 6 = 10 + 0$	$2 + 3 = 1 + 4$
$10 = 4 + 6$	$4 + 6 = 10 + 2$
$10 = 10$	$4 + 6 = 4 + 6$
$4 + 6 = 0 + 10$	$4 + 6 = 6 + 4$

B. Write three of your own true or false equations. Ask your partner to decide if your equations are true or false. Discuss.

C. What numbers will make the following equations true?

$4 + 6 =$ ___ $+ 6$	$28 + 15 =$ ___ $+ 14$
$4 + 7 =$ ___ $+ 8$	$9 +$ ___ $= 8 + 4$
$28 + 3 =$ ___ $+ 2$	$8 =$ ___

need nothing more than a simple explanation. Research has shown, however, that many students lack an adequate understanding of the equal sign. Figure 2 depicts an early algebra activity that focuses on fostering an understanding of the equal sign as representing a relation between two equivalent quantities.

In this activity, students work with the equal sign in a variety of equation formats. The equations are designed to elicit particular student misconceptions (several of which are highlighted below) so teachers can address these in a class discussion. Students' difficulties in understanding the equal sign as a relation between two equivalent quantities are due, in large part, to the computation- and answer-oriented focus of elementary school mathematics as well as to how teachers present number sentences to students (almost always with operations on the left side and answer on the right side: a + b = c).

Due to these experiences, students often view the equal sign as the symbol that precedes the answer or as an indicator to compute what is to its left. These conceptions often lead students to state that an equation such as 2 + 3 = 5 + 4 is true because 2 + 3 = 5 (in such cases, the 4 is not even considered); that 10 = 10 is false because there is "nothing to do;" that 10 = 4 + 6 is backward because the operation is to the right of the equal sign; or that placing an 11 in the blank makes the equation 4 + 7 = ___ + 8 true because 4 + 7 = 11. Many of the difficulties that secondary school algebra students have when working with symbolic expressions and equations may be attributed to their misconceptions about the meaning of the equal sign (Knuth et al., 2006). Yet, when provided the opportunity to work with and discuss equations in a variety of formats, we know that even very young students can develop a relational understanding of the equal sign.

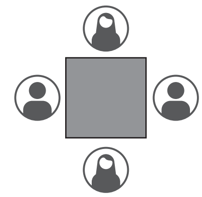
#3. Student success in algebra also requires an ability to detect and generate patterns and to generalize those patterns symbolically. See Figure 3 for an early algebra activity that focuses on developing students' abilities to reason about and express how two quantities vary in relation to each other.

In this activity, students might approach completing the table by drawing the additional desks and people for each case, or by drawing additional desks for several cases and then completing the table after noticing a pattern in the column representing the number of people (i.e., for each new row, the number of people increases by 2). In subsequent parts, students engage in generalizing the relationship between the co-varying quantities (i.e., number of desks and number of people), representing the relationship as a rule using natural language and algebraic notation (e.g., 2x + 2 = y), and using the rule to make a prediction. Although elementary school curricula often include a focus on detecting and generating patterns, the focus is often on one-variable patterns (e.g., identifying the next term and rule for a pattern such as 2, 4, 6, 8). An exclusive focus on this type of activity can hinder the development of students' reasoning about co-varying quantities.

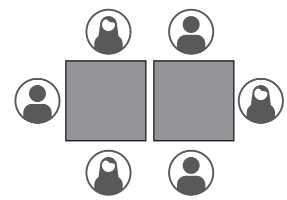
FIGURE 3.

Developing an understanding of functional relationships

A. Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square desks. He can seat four people at one square desk in this way:



If he joins another square desk to the first one, he can seat six people:



1. If Brady keeps joining square desks in this way, how many people can sit at three desks? At four desks? At five desks? Record your responses in a table.
2. Do you see any patterns in the table? Describe them.
3. Find a rule that describes the relationship between the number of desks and the number of people who can sit at the desks. Describe your rule in words.
4. Represent your rule using variables. What do your variables represent?
5. If Brady has 100 desks, how many people can he seat? Show how you got your answer.

It is important to recognize that the preceding examples of early algebra tasks are not meant to be stand-alone activities. Indeed, the activities included here all revolve around several core ways of thinking algebraically: generalizing, representing, justifying, and reasoning with mathematical relationships (Kaput, 2008). Building a foundation for success in algebra in the secondary grades requires providing students with coordinated and long-term opportunities to develop these core ways of thinking algebraically throughout elementary school. Such an approach must intentionally integrate early algebraic concepts and practices throughout the elementary school curriculum.

> **Typical arithmetic-based elementary school mathematics curricula and instruction does little to prepare students for the successful study of algebra in the later grades.**

Early algebra makes a difference

An emerging body of research on early algebra has provided important evidence regarding children's abilities to think algebraically (Cai & Knuth, 2011; Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2008; Lins & Kaput, 2004). Such research has documented that children can successfully develop critical algebraic thinking skills that are foundational to the successful study of algebra in the secondary grades. Most of this work, however, has focused on specific areas of early algebra to the exclusion of others, and has not been longitudinal. Thus, it's an open question whether a comprehensive and sustained early algebra curricular approach makes a difference in students' readiness for algebra in the secondary grades.

Building a foundation for success in algebra in the secondary grades requires providing students with coordinated and long-term opportunities to develop these core ways of thinking algebraically throughout elementary school.

In our current work — Project LEAP (http://algebra.wceruw.org) — we are addressing the aforementioned question and specifically whether children who experience comprehensive and sustained early algebra in grades 3-5 are better prepared for algebra in secondary school than children who have a traditional arithmetic-based experience in grades 3-5. To date, our findings indicate that elementary school children are capable of learning foundational algebraic concepts and skills and that a comprehensive and sustained early algebra experience significantly affects their algebra understanding (Blanton et al., 2015). For example, students in the early algebra classrooms demonstrated the ability to:

- Think relationally about the equal sign;
- Model mathematical situations containing unknown quantities with algebraic expressions and equations;
- Recognize the underlying structure of properties of arithmetic in equations and use such properties to build mathematical arguments;
- Produce and comprehend variable representations of generalized claims; and
- Identify functional relationships and represent them with variable notation.

Our findings also suggest that an early algebra education can potentially eliminate some of the difficulties students have with algebra in the secondary grades. We also are finding that students who experienced a business-as-usual approach to elementary school mathematics demonstrate very little gain in their understanding of foundational algebraic concepts and skills and, in fact, in many cases seem to learn very little algebra overall. As such, the performance of these students highlights that typical arithmetic-based elementary school mathematics curricula and instruction does little to prepare students for the successful study of algebra in the later grades.

The promise of early algebra to increase student success in algebra and, ultimately, greater access to educational and employment opportunities, makes it a worthwhile investment. K

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